The ilc package: Iterative Lee-Carter

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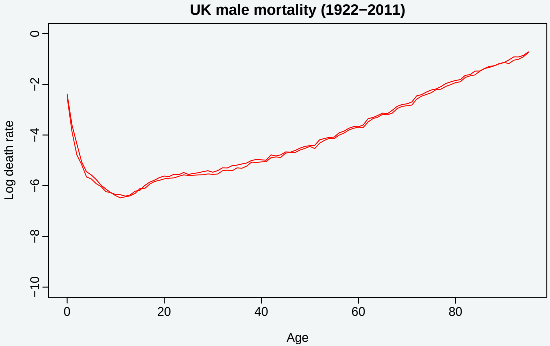
Motivating example: UK male log mortality rates

Force of mortality is

$$\widehat{\mu}_{x,t} = \widehat{m}_{x,t} = \frac{y_{x,t}}{e_{x,t}}$$

where $y_{x,t}$ and $e_{x,t}$ represent the number of deaths and corresponding central exposure for any given age group at year t. Obtain a $n \times p$ matrix to represent age and time dimensions



Lee-Carter model under Gaussian error

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2. κ_t measures the trend in mortality over time
3. β_x measures the age-specific deviations of mortality change from the overall trend
4. $\varepsilon_{x,t}$ are assumed to be $N\left(0, \sigma^2\right)$ random effects by age and time

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- Use Generalised Linear Model (GLM) model (Renshaw and Haberman 2006)
- Develop and implement a stratified LC model for the measurement of the additive effect on the log scale of an explanatory factor (other than age and time)
- Produce forecasts of age-specific mortality rates and life expectancy

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- Instead of singular value decomposition, consider a regression model based on Poisson likelihood maximisation
- ilc package contains methods for the analysis of a class of six log-linear models (capturing age, period, cohort) in the GLM with Poisson errors

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3. Includes two simple methods of “closing-out” produces to correct the original data at very old ages before the application of the model
4. ilc package integrates with the *demography* and *forecast* packages
5. ilc package has improved inspection and graphical visualisation of mortality data and regression output

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$$\mathsf{E}[Y_{x,t}] = e_{x,t}\mu_{x,t}, \quad \mathsf{Var}[Y_{x,t}] = \phi\mathsf{E}[Y_{x,t}]$$

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3. GLM model of the response variable $Y_{x,t}$ with log-link and non-linear parameterized predictor:

$$\eta_{x,t} = \ln(\widehat{y}_{x,t}) = \underbrace{\ln(e_{x,t})}_{\text{offset}} + \alpha_x + \beta_x \kappa_t$$

Estimation algorithm

1. Maximum likelihood point estimates under the GLM approach are obtained at the minimum value of the total deviation, given by

$$D\left(y_{x,t}, \widehat{y}_{x,t}\right) = \sum_{x,t} \mathsf{dev}(x,t) = \sum_{x,t} 2\omega_{x,t} \left\{ y_{x,t} \ln \frac{y_{x,t}}{\widehat{y}_{x,t}} - (y_{x,t} - \widehat{y}_{x,t}) \right\} \quad (1)$$

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   4. Compute $D(y_{x,t}, \widehat{y}_{x,t})$
   5. Repeat the updating cycle; stop when $D(y_{x,t}, \widehat{y}_{x,t})$ converges

Inclusion of cohort effect

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1. Basic LC model can be extended to include an additional bilinear term, containing a second period effect or a cohort effect
2. Force of mortality by a generalised structure is given as

$$\mu_{x,t} = \exp\left(\alpha_x + \underline{\beta_x^{(0)} l_{t-x}} + \beta_x^{(1)} \kappa_t\right)$$

where α_x: main age profile; l_{t-x}: cohort effect; κ_t: period

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2. Consider a cross-classified mortality experience observed over age x, period t and an extra variate g made up of $(k \times n \times l)$ data cells
3. Stratified LC model is given by

$$\eta_{x,t,g} = \ln(\widehat{y}_{x,t,g}) = \underbrace{\ln(e_{x,t,g})}_{\text{offset}} + \alpha_x + \underline{\alpha_g} + \beta_x \kappa_t,$$

where α_g measures the relative differences between the age-specific log mortality profiles among subgroups defined by the extra variate g

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3. Random walk with drift, ARIMA(0,1,0), is used to forecast period effect (κ_t), expressed as

$$\kappa_t = \kappa_{t-1} + d + e_t$$

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4. In the cohort effects, forecasts revert to the fitted parameters when h falls within the available data range

CMI data

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2. Data is made up of observed central exposure and deaths for ages 50-108 from 1983 to 2003

R demo of explanatory plots

1. Plot mortality rates, population-at-risk, death counts for any age group and year

```
>insp.dd(dd.cmi.pens,age=50:80,year=1985:1990)
>insp.dd(dd.cmi.pens,what='pop',age=70:100,year=1988:1993)
>insp.dd(dd.cmi.pens,what='deaths',age=seq(100),
year=1980:2010)
```

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year=1980:2010)
```

2. Produce simple plots (i.e., without legend) of log- or untransformed rates:

```
>plot(dd.cmi.pens)
>plot(dd.cmi.pens,transform=FALSE)
```

1. Produce annotated plots of log or original rates:

```
>plot_dd(dd.cmi.pens, xlim=c(40, 110),
lpar = list(x.int = -0.2, y.int = 0.9, cex = 0.85))
>plot_dd(dd.cmi.pens, year=1985:1995, transform=FALSE)
>plot_dd(dd.cmi.pens, year=1995:1997, transform=FALSE,
lty=1:3, col=1:3)
```

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>plot_dd(dd.cmi.pens, year=1995:1997, transform=FALSE,
lty=1:3, col=1:3)
```
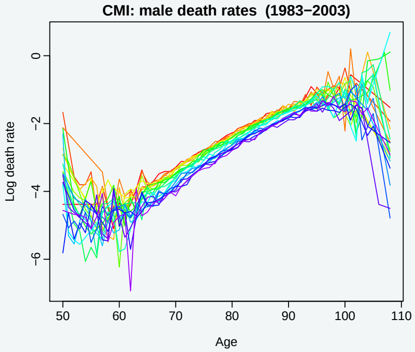
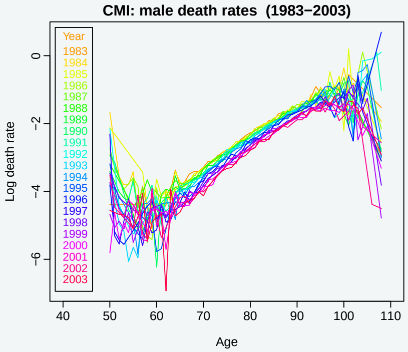

2. Deal with missing data

```
# without correction of empty cells
>tmp.d = extract.deaths(dd.cmi.pens, ages=55:100)
# empty cells are filled using perk model
>tmp.d = extract.deaths(dd.cmi.pens, ages=55:100,
fill='perks')
```

Explanatory plots: dealing with missing values

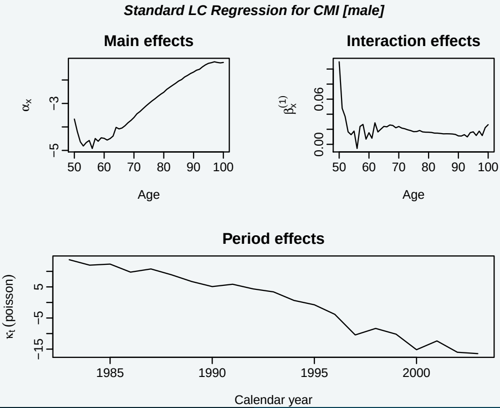




R demo for estimation of LC model

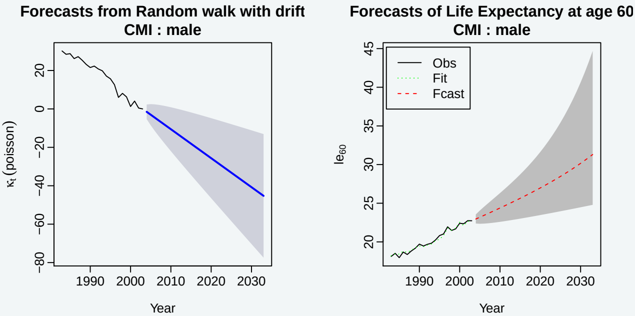
Estimate the base LC model with Poisson errors

```
>mod6 = lca.rh(dd.cmi.pens, mod = 'lc', interpolate=TRUE)
>coef(mod6); plot(mod6)
>fitted_plot(mod6); residual_plot(mod6)
```



Plots of fitted models

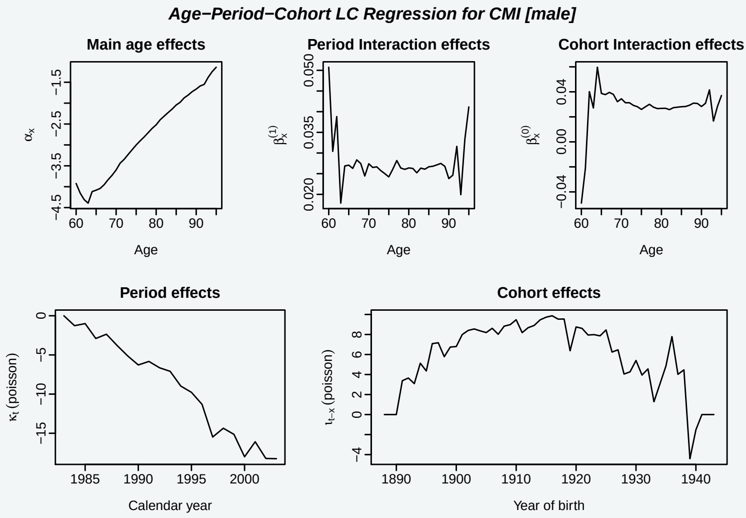
```
>forc6 = forecast(mod6, h = 20, jump = 'fit', level = 90,
shift=FALSE)
>plot_dd(forc6, xlim=c(45,100), lpar=list(x.int=-0.2,
y.int=0.9, cex=0.95))
>le6 = life.expectancy(forc6, age=60)
>flc.plot(mod6, at=60, h=30, level=90)
```



R demo for estimation of age-period-cohort model

```
>mod1 = lca.rh(dd.cmi.pens, age=60:95, mod = "m",
restype='deviance', dec.conv=3)
>coef(mod1)
>plot(mod1)
```

Age-period-cohort plot



R demo for stratified LC model

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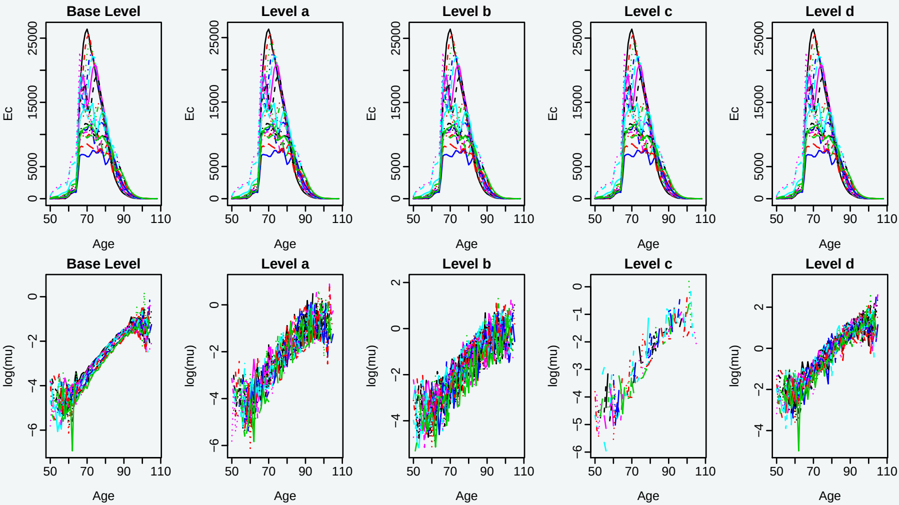
1. For stratified LC model, ilc package introduces a special class of data object that holds information about the grouping factors and aggregate data of number of deaths, central exposures and mortality rates
2. Taking the CMI experience as the base data, produce a randomly stratified mortality data

```
>rfp.cmi = dd.rfp(dd.cmi.pens, rfp = c(0.5,1.2,-0.7,2.5))
>matplot(rfp.cmi$age, rfp.cmi$pop[,,1], type='l',
xlab='Age', ylab='Ec', main = 'Base Level')
>matplot(rfp.cmi$age, rfp.cmi$pop[,,2], type='l',
xlab='Age', ylab='Ec', main = 'Base Level')
```

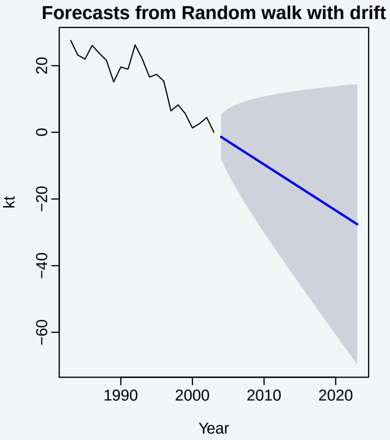
Plots of stratified Lee-Carter

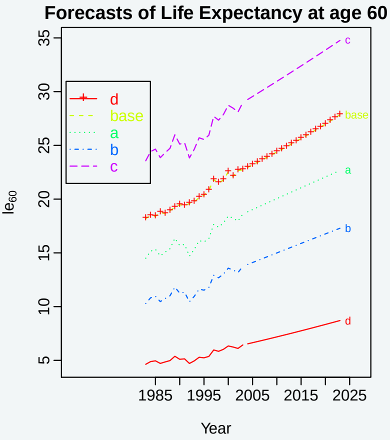


R demo for estimating and forecasting of stratified LC model

```
>rfp.cmi = dd.rfp(dd.cmi.pens, rfp = c(0.5, 1.2, -0.7, 2.5))
>mod6e = elca.rh(rfp.cmi, age=50:100, interpolate=TRUE,
dec.conv=3, verbose=TRUE)
>coef(mod6e)
>mod6ef = forecast.lca(mod6e, h = 20, level = 90, jump=‘fit’,
shift=FALSE)
>plot(mod6ef$kt, ylab=‘kt’, xlab=‘Year’)
>matfle.plot(mod6e$lca, mod6, at=60, label=‘RFP CMI’, h=20)
```

Plot of forecast life expectancy





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2. ilc package implements additional five models discussed in Renshaw and Haberman (2006)
3. Stratified Lee-Carter model allows users to include additional covariates (other than age and time)

References

- Renshaw, A. E., and S. Haberman, 2003. Lee-Carter mortality forecasting with age-specific enhancement. *Insurance: Mathematics and Economics*, 33, 255-272.

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- Hyndman, R. J., and H. L. Shang, 2010. Rainbow plots, bagplots and boxplot for functional data, *Journal of Computational and Graphical Statistics*, 19(1), 29-45.

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- Renshaw, A. E., and S. Haberman, 2003. Lee-Carter mortality forecasting with age-specific enhancement. *Insurance: Mathematics and Economics*, 33, 255-272.
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Thank you! · Tak